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Research article

Conductors of Abhyankar-Moh semigroups of even degrees

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Abstract: In their paper on the embeddings of the line in the plane, Abhyankar and Moh proved an important inequality, now known as the Abhyankar-Moh inequality, which can be stated in terms of the semigroup associated with the branch at infinity of a plane algebraic curve. Barrolleta, García Barroso and Płoski studied the semigroups of integers satisfying the Abhyankar-Moh inequality and call them Abhyankar-Moh semigroups. They described such semigroups with the maximum conductor. In this paper we prove that all possible conductor values are achieved for the Abhyankar-Moh semigroups of even degree. Our proof is constructive, explicitly describing families that achieve a given value as its conductor.

Keywords: Abhyankar-Moh inequality; Abhyankar-Moh semigroups; conductor; gluing of semigroups; branch at infinity

1. Introduction

Suzuki [1] and Abhyankar and Moh [2] proved independently that the affine line can be embedded in a unique way, up to ambient automorphisms, in the affine plane. Let us indicate some details of this fact. Let K be an algebraically closed field of arbitrary characteristic. A polynomial mapping $\sigma_{p,q} : K \longrightarrow K^2$ defined as $\sigma_{p,q}(x, y) = (p(x, y), q(x, y))$ is a *polynomial embedding of the line* K if there is a polynomial map $g : K^2 \longrightarrow K$ such that $g(p(t), q(t)) = t$ in $K[t]$. This is equivalent to the equality $K[p(t), q(t)] = K[t]$.

An affine curve $C \subset K^2$ is an *embedded line* if there exists a polynomial embedding $\sigma_{p,q}$ such that $\sigma_{p,q}(K) = C$. It is easy to check that any embedded line is an irreducible affine curve.

Let C be an embedded line with a minimal equation $f(x, y) = 0$. After [2], the curve C has only *one place at infinity*, that is, the closure $\overline{C}$ of C in the projective plane has only one point O_∞ on the line at infinity, and it is unibranch at O_∞ (the polynomial $f(x, y)$ is irreducible as an element of the formal power series ring $K[[x, y]]$). In this case, associated with $\overline{C}$ we have a numerical semigroup $S(\overline{C})$ consisting of zero and all intersection numbers of $\overline{C}$ with all algebroid curves not having $\overline{C}$ as an irreducible component. By the Bresinsky-Angermüller Theorem ([3] for zero characteristic and [4] for arbitrary characteristic) there exists a (unique) sequence $(v_0, \ldots, v_h)$, called the *characteristic at infinity* of C, generating $S(\overline{C})$ where v_0 is the degree of C.
Assume that C is an affine irreducible curve of degree greater than 1 with one branch at infinity and let $(v_0, \ldots, v_h)$ be its characteristic at infinity. Suppose that $\gcd(\deg C, \mathrm{ord}_{O_\infty}\overline{C}) \not\equiv 0 \pmod{\mathrm{char}\, K}$. Then, after [2]

$$\gcd(v_0, \ldots, v_{h-1})v_h < v_0^2. \tag{1.1}$$

The condition $\gcd(\deg C, \mathrm{ord}_{O_\infty}\bar{C}) \not\equiv 0 \pmod{\mathrm{char}\, K}$ is automatically satisfied when the characteristic of K is zero, but otherwise it is essential.

The inequality (1.1) is called the Abhyankar-Moh inequality. Originally, this inequality appears linked to the Puiseux expansion of the given branch at the infinite place (see [2, equality (35)]). The semigroups of integers associated with branches and satisfying the Abhyankar-Moh inequality are called *Abhyankar-Moh semigroups* of degree v_0 (the order of the branch). The Abhyankar-Moh semigroups were studied in [5], where these semigroups with maximum possible conductor, which is equal to $(v_0 - 2)(v_0 - 1)$, were described. Later, in [6], a geometric interpretation of the branches with Abhyankar-Moh semigroup and maximum possible conductor was given. It is well known that the conductor of an Abhyankar-Moh semigroup of degree v_0 is an even integer belonging to the interval $[v_0 - 1, (v_0 - 1)(v_0 - 2)]$. Our main result is the following:

Theorem A. *Let $n > 2$ be an even natural number. For any even number c with $n-1 \le c \le (n-1)(n-2)$, there is an Abhyankar-Moh semigroup of degree n and conductor equal to c.*

In order to prove Theorem A, we start by computing in Section 3, in an algorithmic way, the set of all Abhyankar-Moh semigroups of a fixed degree (see Algorithm 2). Section 4 is devoted to the proof of Theorem A. This proof is constructive, and the algorithms of Section 3 play a fundamental role in it.

From the computational point of view, it remains, as an open question, to determine the values of conductors reached by the Abhyankar-Moh semigroups of odd degree, but it seems that this poses new computational challenges. From the geometric point of view, the next step would be to geometrically characterize the branches with Abhyankar-Moh semigroups of even degree, studied in this paper, following the line given in [6] for those with the maximum possible conductor.

2. Preliminaries

A numerical semigroup S is an additive submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$. It is well known that numerical semigroups are finitely generated, and their minimal generating sets are unique. The largest integer in $\mathbb{N} \setminus S$ is called the *Frobenius number* of S. The *conductor* of S is the Frobenius number of S plus 1.

Given a finite set $A = \{a_1, \ldots, a_t\} \subset \mathbb{N}$, A is the generating set of S when $S = \mathbb{N}a_1 + \cdots + \mathbb{N}a_t$. In this work, when A is the minimal generating set of S, we assume that $a_1 < \cdots < a_t$. When we write $S = \langle A \rangle$, we imply that A is the minimal set of generators of S. The cardinality of the minimal generating set of S is called the *embedding dimension* of S.

A sequence of positive integers $(v_0, \ldots, v_h)$ is called a *characteristic sequence* if it satisfies the following two properties:

(CS1) Put $e_k = \gcd(v_0, \ldots, v_k)$ for $0 \le k \le h$. Then, $e_k < e_{k-1}$ for $1 \le k \le h$ and $e_h = 1$.
(CS2) $e_{k-1}v_k < e_k v_{k+1}$ for $1 \le k \le h-1$.

We put $n_k = \frac{e_{k-1}}{e_k}$ for $1 \le k \le h$. Therefore, $n_k > 1$ for $1 \le k \le h$ and $n_h = e_{h-1}$. If $h = 0$, the only characteristic sequence is $(v_0) = (1)$. If $h = 1$, the sequence (v_0, v_1) is a characteristic sequence if and only if $\gcd(v_0, v_1) = 1$ and $v_0 > 1$. Property (CS2) plays a role if and only if $h \ge 2$.

Lemma 1. *([5, Lemma 1.1]) Let $(v_0, \ldots, v_h)$ be a characteristic sequence with $h \ge 2$. Then,*

(i) $v_1 < \cdots < v_h$, *and* $v_0 < v_2$.
(ii) *Let* $v_1 < v_0$. *If* $v_0 \not\equiv 0, \pmod{v_1}$, *then* $(v_1, v_0, v_2, \ldots, v_h)$ *is a characteristic sequence. If* $v_0 \equiv 0, \pmod{v_1}$ *then* $(v_1, v_2, \ldots, v_h)$ *is a characteristic sequence.*

A semigroup $S \subseteq \mathbb{N}$ is *strongly increasing* (SI-semigroup) if $S \neq \{0\}$ and it is generated by a characteristic sequence, that is, $S = \mathbb{N}v_0 + \cdots + \mathbb{N}v_h$. We will denote by $S = S(v_0, v_1, \ldots, v_h)$ the numerical semigroup generated by the characteristic sequence $(v_0, v_1, \ldots, v_h)$.

A SI-semigroup $S(v_0, v_1, \ldots, v_h) \subseteq \mathbb{N}$ is an *Abhyankar-Moh semigroup (A-M semigroup)* of degree $n = v_0 > 1$ if it satisfies the *Abhyankar-Moh inequality*

$$e_{h-1}v_h < n^2.$$

Observe that the semigroup $\mathbb{N} = S(n, 1)$ for any $n \in \mathbb{N}$, so $\mathbb{N}$ is an Abhyankar-Moh semigroup of any degree.

The conductor of the A-M semigroup $S = S(v_0, v_1, \ldots, v_h)$ is

$$c(S) = \sum_{i=1}^{h}(n_i - 1)v_i - v_0 + 1, \tag{2.1}$$

where $n_i = \frac{e_{i-1}}{e_i}$ for $i \in \{1, \ldots, h\}$. Moreover $c(S)$ is an even integer (see [5, Proposition 1.2]).

Remark 2. *If $S = S(v_0, v_1, \ldots, v_h)$ is an A-M semigroup of degree $n = v_0 > 1$, then $v_0 > v_1$, since $e_0v_1 < e_1v_2 < \cdots < e_{h-1}v_h < n^2 = v_0^2$.*

Let $S = S(v_0, v_1, \ldots, v_h)$ be an A-M semigroup of degree $n = v_0 > 1$. If $c(S)$ is the conductor of S, then, by [5, Theorem 2.2],

$$c(S) \le (n-1)(n-2). \tag{2.2}$$

Moreover,

$$c(S) = (n-1)(n-2) \text{ if and only if } v_k = \frac{n^2}{e_{k-1}} - e_k \text{ for } 1 \le k \le h. \tag{2.3}$$

Hence, if $S \neq \mathbb{N}$ is an A-M semigroup of degree $n > 1$, its conductor $c(S)$ is an even integer satisfying the inequalities

$$n - 1 \le c(S) \le (n-1)(n-2). \tag{2.4}$$

By Remark 2 we get that the only A-M semigroup of degree 2 is generated by the characteristic sequence $(2, 1)$. Such a semigroup achieves the upper bound for the conductor, given in (2.2).

After [5, Proposition 1.2], if $(v_0, v_1, \ldots, v_h)$ is a characteristic sequence and $S = S(v_0, v_1, \ldots, v_h)$, then $\{\min(v_0, v_1), v_2, \ldots, v_{h-1}, v_h\}$ is a subset of the minimal generating set of S, and $v_2 \notin \mathbb{N}v_0 + \mathbb{N}v_1$. Furthermore, for every $w \in S$, $(v_0, v_1, \ldots, v_i, w, v_{i+1}, \ldots, v_h)$ is not a characteristic sequence, for every $i = 0, \ldots, h-1$.

Let $S = S(v_0, v_1, \ldots, v_h)$ be a SI-semigroup. By Lemma 1, the characteristic sequences generating S are the following:

- $(v_0, v_1, v_2, \ldots, v_h)$.
- If $v_0 < v_1$, $(v_1, v_0, v_2, \ldots, v_h)$.
- If $v_1 < v_0$ and $v_0 \not\equiv 0 \pmod{v_1}$, $(v_1, v_0, v_2, \ldots, v_h)$.
- If $v_1 < v_0$ and $v_0 \equiv 0 \pmod{v_1}$, $(v_1, v_2, \ldots, v_h)$ and $(v_2, v_1, \ldots, v_h)$.
- $(kv'_0, v'_0, v'_1, v_2, \ldots, v_h)$ for every integer $k \in [2, v'_1/v'_0]$ where $v'_0 = \min(v_0, v_1)$, $v'_1 = \max(v_0, v_1)$, and $v'_1 \not\equiv 0 \pmod{v'_0}$.

As a consequence, we determine the characteristic sequences generating a SI-semigroup given by its minimal generating set.

Let $S = \langle a_1, \ldots, a_t \rangle$ be a SI-semigroup. Then, from [5, Corollary 1.4], the characteristic sequences generating S are the following:

$$\begin{aligned} &\bullet\, (a_1, \ldots, a_t), \\ &\bullet\, (a_2, a_1, a_3, \ldots, a_t) \\ &\bullet \text{ and } (ka_1, a_1, a_2, \ldots, a_t) \text{ for every integer } k \in [2, a_2/a_1). \end{aligned} \tag{2.5}$$

Similarly, the possible characteristic sequences generating an A-M semigroup given by its minimal generating set can be described: Let $S = \langle a_1, \ldots, a_t \rangle$ be a SI-semigroup. If S is an A-M semigroup, then

$$S = S(a_2, a_1, a_3, \ldots, a_t), \text{ or } S = S(ka_1, a_1, a_2, \ldots, a_t) \tag{2.6}$$

for every integer $k \in \left(\frac{\sqrt{\gcd(a_1, \ldots, a_{t-1}) a_t}}{a_1}, \frac{a_2}{a_1} \right)$. For the first case, S should be an A-M semigroup of degree a_2, and it should be of degree ka_1 for the second (see [5, Proposition 2.1]).

Notice that if you want to check whether the semigroup $\langle a_1, \ldots, a_t \rangle$ is an A-M semigroup, you only need to check whether it is an A-M semigroup of degree a_2.

3. Computing Abhyankar-Moh semigroups

The A-M semigroups minimally generated by two elements are easy to describe. Whenever $b > a > 1$ are two coprime integers, $\langle a, b \rangle = S(b, a)$ is an A-M semigroup of degree b. Furthermore, $\langle a, b \rangle = S(ka, a, b)$ is also an A-M semigroup of degree ka for every $k \in (\sqrt{b/a}, b/a) \cap \mathbb{N}$.

A-M semigroups of higher embedding dimensions are completely characterized by *gluings*: The gluing of $S = \langle a_1, \ldots, a_t \rangle$ and $\mathbb{N}$ with respect to the positive integers d and f with $\gcd(d, f) = 1$ (see [7, Chapter 8]) is the numerical semigroup $\mathbb{N}da_1 + \cdots + \mathbb{N}da_t + \mathbb{N}f$. We denote it by $S \oplus_{d,f} \mathbb{N}$.

Proposition 3. *The set $\bar{S}$ is an A-M semigroup with embedding dimension $t \geq 3$ if and only if $\bar{S} = S \oplus_{d,f} \mathbb{N}$ where $S = \langle b_1, \ldots, b_{t-1} \rangle$ is an A-M semigroup of degree m, and f, d are two coprime integers such that $dm^2 > f > d \gcd(b_1, \ldots, b_{t-2}) b_{t-1}$. Moreover, the degree of $\bar{S}$ is dm.*

Proof. Consider $\bar{S} = \langle a_1, \ldots, a_t \rangle$. Put $d = \gcd(a_1, \ldots, a_{t-1})$, $f = a_t$, $b_i = a_i/d$ for all $i = 1, \ldots, t-1$, and the SI-semigroup $S = \langle b_1, \ldots, b_{t-1} \rangle$. Trivially, $\bar{S} = S \oplus_{d,f} \mathbb{N}$. Since $\bar{S}$ is a SI-semigroup,

$$\begin{aligned} f &= a_t > \gcd(a_1, \ldots, a_{t-2})a_{t-1}/d = d \gcd(a_1/d, \ldots, a_{t-2}/d)a_{t-1}/d \\ &= d \gcd(b_1, \ldots, b_{t-2})b_{t-1}. \end{aligned}$$

By (2.6), the degree of $\bar{S}$ equals a_2, or ka_1 for some integer $k > 1$. If we assume that the degree is a_2, then $(a_2, a_1, a_3, \ldots, a_t)$ is a characteristic sequence generating $\bar{S}$, and $a_2^2 > \gcd(a_1, \ldots, a_{t-1})a_t$. Thus, $(b_2, b_1, b_3, \ldots, b_{t-1})$ is a characteristic sequence generating S, and

$$\begin{aligned} b_2^2 &= (a_2/d)^2 > \gcd(a_1, \ldots, a_{t-1})a_t/d^2 > \gcd(a_1, \ldots, a_{t-2})a_{t-1}/d^2 \\ &= \gcd(b_1, \ldots, b_{t-2})b_{t-1}. \end{aligned}$$

Hence, S is an A-M semigroup of degree $m = b_2$.

Similarly, for the degree ka_1, $(kb_1, b_1, b_2, \ldots, b_{t-1})$ is a characteristic sequence generating S, and $(kb_1)^2 = (ka_1/d)^2 > \gcd(b_1, \ldots, b_{t-2})b_{t-1}$, so S is an A-M semigroup of degree $m = kb_1$. For both of the previous possibilities, the inequality $dm^2 > f$ is satisfied.

Conversely, let $S = \langle b_1, \ldots, b_{t-1} \rangle$ be an A-M semigroup of degree $m \in \mathbb{N}$ and f, d be two coprime integers such that $dm^2 > f > d \gcd(b_1, \ldots, b_{t-2})b_{t-1}$. Since $f > d \gcd(b_1, \ldots, b_{t-2})b_{t-1}$, $\bar{S} = S \oplus_{d,f} \mathbb{N} = \langle db_1, \ldots, db_{t-1}, f \rangle$ is a SI-semigroup ([8, Theorem 3]). Again, by (2.6), the possible characteristic sequences generating S are $(b_2, b_1, b_3, \ldots, b_{t-1})$ and $(kb_1, b_1, b_3, \ldots, b_{t-1})$, with degrees b_2 and kb_1, respectively. Hence, the characteristic sequences generating $\bar{S}$ are $(db_2, db_1, db_3, \ldots, db_t, f)$ and $(kdb_1, db_1, db_2, \ldots, db_t, f)$; and by hypothesis, we have the Abhyankar-Moh inequality $(dm)^2 > df$. Thus, $\bar{S}$ is an A-M semigroup of degree dm. □

Let S be an A-M semigroup of degree m, and $f > d > 1$ are two coprime integers such that $S \oplus_{d,f} \mathbb{N}$ is also an A-M semigroup of degree dm. Then, by [9, Proposition 10],

$$\mathrm{c}(S \oplus_{d,f} \mathbb{N}) = d\mathrm{c}(S) + (d-1)(f-1), \tag{3.1}$$

where $\mathrm{c}(S)$ denotes the conductor of S.

Denote by $M(A)$ the largest element of the minimal system of generators of the numerical semigroup $\langle A \rangle$, and $s(A) = \min(A \setminus \{\min(A)\})$, that is, the second element in A. Algorithm 1 computes all the A-M semigroups with conductor less than or equal to a fixed non-negative integer.

Table 1 illustrates Algorithm 1: We collect all the A-M semigroups with conductor less than or equal to 18. We also give the characteristic sequences associated with the given semigroups.

Algorithm 1: Computation of the set of A-M semigroups with conductor less than or equal to c.

Input: $c \in \mathbb{N} \setminus \{0, 1\}$.
Output: The set $\{A \mid \langle A \rangle$ is an A-M semigroup with $\mathrm{c}(\langle A \rangle) \leq c\}$.

$\mathcal{A} \leftarrow \{\{a, b\} \mid 1 < a < b,\ \gcd(a, b) = 1,\ ab - a - b + 1 \leq c\}$;
forall $k \in \{2, \ldots, c - 1\}$ **do**
  $B \leftarrow \{A \in \mathcal{A} \mid \mathrm{c}(\langle A \rangle) = k\}$;
  forall $A \in B$ **do**
    $G_A \leftarrow \{(d, f) \in \mathbb{N}^2 \mid \gcd(f, d) = 1,\ s(A)^2 > f/d > \gcd\left(A \setminus \{M(A)\}\right) \cdot M(A),\ c \geq \max\{dk + (d - 1)(f - 1), d^2(k - 1) + 1\}\}$;
  $\mathcal{A} \leftarrow \mathcal{A} \cup \{dA \cup \{f\} \mid (d, f) \in G_A\}$;
return $\mathcal{A}$;

Table 1. A-M semigroups up to conductor 18.

A-M semigroups	Characteristic sequences
$\langle 2, 3 \rangle$	$\{(3, 2)\}$
$\langle 2, 5 \rangle$	$\{(4, 2, 5), (5, 2)\}$
$\langle 2, 7 \rangle$	$\{(4, 2, 7), (6, 2, 7), (7, 2)\}$
$\langle 2, 9 \rangle$	$\{(6, 2, 9), (8, 2, 9), (9, 2)\}$
$\langle 2, 11 \rangle$	$\{(6, 2, 11), (8, 2, 11), (10, 2, 11), (11, 2)\}$
$\langle 2, 13 \rangle$	$\{(6, 2, 13), (8, 2, 13), (10, 2, 13), (12, 2, 13), (13, 2)\}$
$\langle 2, 15 \rangle$	$\{(6, 2, 15), (8, 2, 15), (10, 2, 15), (12, 2, 15), (14, 2, 15), (15, 2)\}$
$\langle 2, 17 \rangle$	$\{(6, 2, 17), (8, 2, 17), (10, 2, 17), (12, 2, 17), (14, 2, 17), (16, 2, 17), (17, 2)\}$
$\langle 2, 19 \rangle$	$\{(8, 2, 19), (10, 2, 19), (12, 2, 19), (14, 2, 19), (16, 2, 19), (18, 2, 19), (19, 2)\}$
$\langle 3, 4 \rangle$	$\{(4, 3)\}$
$\langle 3, 5 \rangle$	$\{(5, 3)\}$
$\langle 3, 7 \rangle$	$\{(6, 3, 7), (7, 3)\}$
$\langle 3, 8 \rangle$	$\{(6, 3, 8), (8, 3)\}$
$\langle 3, 10 \rangle$	$\{(6, 3, 10), (9, 3, 10), (10, 3)\}$
$\langle 4, 5 \rangle$	$\{(5, 4)\}$
$\langle 4, 7 \rangle$	$\{(7, 4)\}$
$\langle 4, 6, 13 \rangle$	$\{(6, 4, 13)\}$
$\langle 4, 6, 15 \rangle$	$\{(6, 4, 15)\}$
$\langle 4, 6, 17 \rangle$	$\{(6, 4, 17)\}$

Let $n > 1$ be an integer. A sequence of integers $(d_0, \ldots, d_h)$ will be called a *sequence of divisors of* n if d_i divides d_{i-1} for $1 \leq i \leq h$ and $n = d_0 > d_1 > \cdots > d_{h-1} > d_h = 1$. In particular, if $(v_0, v_1, \ldots, v_h)$ is a characteristic sequence, then $(e_0, \ldots, e_h)$ is a sequence of divisors of $n = v_0$, where $e_i = \gcd(v_0, \ldots, v_i)$.

In this case we will say that $(e_0, \ldots, e_h)$ is the sequence of divisors associated with $(v_0, \ldots, v_h)$.

Using sequences of divisors, in [5, Proposition 2.3] it was proved that

$$\left(n, n-d_1, \frac{n^2}{d_1}-d_2, \ldots, \frac{n^2}{d_{i-1}}-d_i, \ldots, \frac{n^2}{d_{k-1}}-1\right) \tag{3.2}$$

is a characteristic sequence, and the semigroup generated by it is an A-M semigroup of degree n and conductor $(n-1)(n-2)$ (which is the maximal possible conductor after (2.2)). Inspired by this idea, using sequences of divisors, we introduce an algorithm for computing all the A-M semigroups for a given degree (Algorithm 2). The following proposition is the key for providing this algorithm.

Proposition 4. *Let $n \geq 2$ be an integer and $D = (d_0, \ldots, d_h)$ be a sequence of divisors of $n = d_0$. Then, the characteristic sequence of any A-M semigroup with degree n and sequence of divisors equal to D is of the form $(n, d_1k_1, \ldots, d_hk_h)$, with $1 \leq k_1 \leq \frac{d_0}{d_1}-1$, $d_{i-2}k_{i-1}+1 \leq d_ik_i \leq \frac{d_0^2}{d_{i-1}}-d_i$ for any $i = 2, \ldots, h$, and $\gcd\left(\frac{d_{i-1}}{d_i}, k_i\right) = 1$ for $i = 1, \ldots, h$.*

Proof. Notice that the condition $\gcd\left(\frac{d_{i-1}}{d_i}, k_i\right) = 1$ for $i = 1, \ldots, h$ guarantees that the characteristic sequence $(n, d_1k_1, \ldots, d_hk_h)$ has associated sequence of divisors equal to D. It remains to check the bounds on the k_i. The case $h = 1$ is trivially verified. Suppose $h \geq 2$. Let $v_0 = d_0$ and $v_i = d_ik_i$ for any $i = 1, \ldots, h$, and assume that $(v_0, \ldots, v_h)$ is the characteristic sequence of an A-M semigroup S of degree $n = v_0$ and associated sequence of divisors $(d_0, \ldots, d_h)$. Since A-M semigroups are SI-semigroups, by the definition of v_i, $d_{i-1}k_i < d_{i+1}k_{i+1}$ for all $i \in \{1, \ldots, h-1\}$. Hence, $d_{i-1}k_i + 1 \leq d_{i+1}k_{i+1}$. Moreover, since $1 < v_1 < v_0$, $1 \leq k_1 \leq \frac{d_0}{d_1}-1$; and since $d_{h-1}k_h < d_0^2$ and $k_h \leq \frac{d_0^2}{d_{h-1}}-1$, this is enough to finish the proof when $h = 2$. Consider now that $h \geq 3$. By Proposition 3, $S = S_{h-1} \oplus_{d_{h-1},k_h} \mathbb{N}$ where S_{h-1} is the A-M semigroup of degree $\frac{d_0}{d_{h-1}}$ generated by the characteristic sequence $\left(\frac{d_0}{d_{h-1}}, \frac{d_1}{d_{h-1}}k_1, \ldots, \frac{d_{h-2}}{d_{h-1}}k_{h-2}, k_{h-1}\right)$. So, $\frac{d_{h-2}}{d_{h-1}}k_{h-1} < \left(\frac{d_0}{d_{h-1}}\right)^2$. Then $k_{h-1} \leq \frac{d_0^2}{d_{h-1}d_{h-2}}-1$, and hence $d_{h-1}k_{h-1} \leq \frac{d_0^2}{d_{h-2}}-d_{h-1}$. In general, using this process, for any $i \in \{3, \ldots, h-1\}$, $S_i = S_{i-1} \oplus_{d_{i-1},k_i} \mathbb{N}$, where S_{i-1} is the A-M semigroup of degree $\frac{d_0}{d_{i-1}}$ generated by the characteristic sequence $\left(\frac{d_0}{d_{i-1}}, \frac{d_1}{d_{i-1}}k_1, \ldots, \frac{d_{i-2}}{d_{i-1}}k_{i-2}, k_{i-1}\right)$. For any i, the condition that S_{i-1} is an A-M semigroup implies that $\frac{d_{i-2}}{d_{i-1}}k_{i-1} < (\frac{d_0}{d_{i-1}})^2$, and $d_{i-1}k_{i-1} \leq \frac{d_0^2}{d_{i-2}}-d_{i-1}$. □

Algorithm 2 provides a computational method to compute all A-M semigroups with fixed degree. Note that this algorithm supports parallel deployment.

Table 2 shows all the A-M semigroups with degree eight. Note that, in this example, all the even integers in $[n-1, (n-1)(n-2)]$ are the conductor of some A-M semigroup of degree $n = 8$. In this work, we prove that this is true for all even degrees.

If $n > 2$, $(d_0, \ldots, d_h = 1)$ is a sequence of divisors of degree $n = d_0$, and we consider values k_i as in Proposition 4, then, by (2.1), the conductor of the A-M semigroup $S(n, d_1k_1, \ldots, d_hk_h)$ is

$$c(S(n, d_1k_1, \ldots, d_hk_h)) = \sum_{i=1}^{h}(d_{i-1}-d_i)k_i - n + 1. \tag{3.3}$$

Note that the characteristic sequence given in (3.2) is $(n, d_1k_1, \ldots, d_hk_h)$ for the maximum values of k_i obtained in Proposition 4. Moreover, as a consequence of that, we have the following corollary.

Algorithm 2: Computation of the set of A-M semigroups with degree n.

Input: $n \in \mathbb{N} \setminus \{0, 1\}$.
Output: The set of the characteristic sequences of A-M semigroups with degree n.

$\mathcal{F}_n \leftarrow \emptyset$;
$\mathcal{D} \leftarrow \{D = (d_0, \dots, d_h) \mid h \in \mathbb{N},\ D$ is a sequence of divisors of $n\}$;
while $\mathcal{D} \neq \emptyset$ **do**
  $D = (d_0, \dots, d_h) \leftarrow \mathrm{First}(\mathcal{D})$;
  $\mathcal{K} \leftarrow \Big\{(k_1, \dots, k_h) \in \mathbb{N}^h \mid \gcd(k_1, n) = \cdots = \gcd(k_h, n) = 1,\ 1 \leq k_1 \leq \frac{d_0}{d_1} - 1$, and $d_{i-2}k_{i-1} + 1 \leq d_i k_i \leq \frac{d_0^2}{d_{i-1}} - d_i$ for any $i = 2, \dots, h\Big\}$;
  forall $(k_1, \dots, k_h) \in \mathcal{K}$ **do**
    $\mathcal{F}_n \leftarrow \mathcal{F}_n \cup \{(n, d_1k_1, \dots, d_hk_h)\}$;
  $\mathcal{D} \leftarrow \mathcal{D} \setminus \{D\}$;
return $\mathcal{F}_n$;

Corollary 5. *Let $n > 2$ be a natural number. Fix a sequence of divisors $(d_0 = n, d_1, \dots, d_h)$ of n. The A-M semigroup of the form $S(n, d_1k_1, \dots, d_hk_h)$ having the minimum conductor is given by $k_1 = 1$ and $k_i = \frac{d_{i-2}}{d_i}k_{i-1} + 1$, for $i \in \{2, \dots, h\}$, and its conductor is $\sum_{i=0}^{h-2} d_i(d_{i+1} - 1)$.*

Proof. By (3.3) and according to Proposition 4, the minimum value of conductors of A-M semigroups of the form $S(n, d_1k_1, \dots, d_hk_h)$ holds for the minimum values of k_i for $i = 1, \dots, h$.

For $h = 1$, the sequence of divisors $(d_0, d_1, \dots, d_h)$ is $(n, 1)$. So, $S(n, 1) = \mathbb{N}$, and its conductor is zero, which is equal to the empty sum $\sum_{i=0}^{-1} d_i(d_{i+1} - 1)$.

Assume that $h \geq 2$. Put $k_1 = 1$, and for any $i \in \{2, \dots, h\}$, we put $k_i = \left\lceil \frac{d_{i-2}k_{i-1}+1}{d_i} \right\rceil = \frac{d_{i-2}k_{i-1}}{d_i} + 1$. Notice that, since $\frac{d_{i-1}}{d_i}$ is a divisor of $\frac{d_{i-2}}{d_i}$, we have $\gcd\left(\frac{d_{i-1}}{d_i}, \frac{d_{i-2}k_{i-1}}{d_i} + 1\right) = 1$. Thus, the integers $k_1 = 1$ and $k_i = \frac{d_{i-2}}{d_i}k_{i-1} + 1$, for $i \in \{2, \dots, h\}$, satisfy Proposition 4; and then, $S(n, d_1k_1, \dots, d_hk_h)$ is the A-M semigroup having the minimum conductor for the sequence of divisors $(d_0 = n, d_1, \dots, d_h)$.

Note that, for $k_1 = 1$ and $k_q = \frac{d_{q-2}k_{q-1}}{d_q} + 1$, we have $k_2 = \frac{d_0}{d_2} + 1$, and, in general, $k_q = \frac{1}{d_{q-1}d_q} \sum_{i=0}^{q-3} d_i d_{i+1} + \frac{d_{q-2}}{d_q} + 1$ for $q \in \{2, \dots, h\}$.

After (3.3), we get $c(S(n, d_1k_1, \dots, d_hk_h)) = \sum_{i=1}^{h}(d_{i-1} - d_i)k_i - n + 1$. Thus,

$$\begin{aligned}
c(S(n, d_1k_1, \dots, d_hk_h)) &= \sum_{i=1}^{h} d_{i-1}k_i - \sum_{i=1}^{h} d_ik_i - n + 1 \\
&= \sum_{i=1}^{h} d_{i-1}k_i - \sum_{i=2}^{h} d_ik_i - d_1 - n + 1 \\
&= \sum_{i=1}^{h} d_{i-1}k_i - \sum_{i=2}^{h} d_i\left(\frac{d_{i-2}}{d_i}k_{i-1} + 1\right) - d_1 - n + 1 \\
&= d_{h-1}k_h - \sum_{i=0}^{h} d_i + 1
\end{aligned}$$

$$
\begin{aligned}
&= d_{h-1}\left(\frac{1}{d_{h-1}}\sum_{i=0}^{h-3} d_i d_{i+1} + d_{h-2} + 1\right) - \sum_{i=0}^{h} d_i + 1 \\
&= \sum_{i=0}^{h-2} d_i d_{i+1} - \sum_{i=0}^{h-2} d_i = \sum_{i=0}^{h-2} d_i(d_{i+1} - 1).
\end{aligned}
$$

□

Table 2. All the A-M semigroups of degree 8.

Characteristic sequences	Sequences of divisors	Conductors
(8,2,9)	(8,2,1)	8
(8,2,11)	(8,2,1)	10
(8,2,13)	(8,2,1)	12
(8,3)	(8,1)	14
(8,2,15)	(8,2,1)	14
(8,2,17)	(8,2,1)	16
(8,2,19)	(8,2,1)	18
(8,2,21)	(8,2,1)	20
(8,2,23)	(8,2,1)	22
(8,4,9)	(8,4,1)	24
(8,2,25)	(8,2,1)	24
(8,2,27)	(8,2,1)	26
(8,4,10,21)	(8,4,2,1)	28
(8,5)	(8,1)	28
(8,2,29)	(8,2,1)	28
(8,4,10,23)	(8,4,2,1)	30
(8,4,11)	(8,4,1)	30
(8,2,31)	(8,2,1)	30
(8,4,10,25)	(8,4,2,1)	32
(8,4,10,27)	(8,4,2,1)	34
(8,6,25)	(8,2,1)	36
(8,4,10,29)	(8,4,2,1)	36
(8,4,13)	(8,4,1)	36
(8,6,27)	(8,2,1)	38
(8,4,10,31)	(8,4,2,1)	38
(8,6,29)	(8,2,1)	40
(8,4,14,29)	(8,4,2,1)	40
(8,6,31)	(8,2,1)	42
(8,4,14,31)	(8,4,2,1)	42
(8,7)	(8,1)	42
(8,4,15)	(8,4,1)	42

Remark 6. *Given an integer $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t} \geq 2$ with $p_1 > p_2 > \cdots > p_t$ prime integers, the maximum length of any sequence of divisors of n is $\Lambda(n) = \sum_{i=1}^{t} \alpha_i$. Moreover, if we fix h, a length of the sequences of divisors of n, then it allows us to provide a lower bound for the integers which could be realizable as the conductor of an A-M semigroup of degree n with an associated sequence of divisors with length greater than or equal to h: Let us consider that with $c_h := \min\{\sum_{i=0}^{h-2} d_i(d_{i+1} - 1) \mid (d_0, d_1, \ldots, d_h)$ is a sequence of divisors of $n\}$, if $c < c_h$, then c is not realizable as the conductor of an A-M semigroup of degree n with an associated sequence of divisors with length greater than or equal to h.*

Fix $n \in \mathbb{N}$, and let T_h be the set of sequences of divisors of n with length h. Consider two sequences $D, D' \in \cup_{h=1}^{\Lambda(n)} T_h$. If D and D' have different lengths, we add zeros to the end of the one with the smallest length so that D and D' have the same length ℓ, and we can compare D and D' with the lexicographical order $<_{\text{lex}}$ as elements of $\mathbb{N}^\ell$. We have

Proposition 7. *Let $n \in \mathbb{N}$ and $D = (d_0 = n, d_1, \ldots, d_h), D' = (d'_0 = n, d'_1, \ldots, d'_{h'}) \in \cup_{h=1}^{\Lambda(n)} T_h$ with $D <_{\text{lex}} D'$. Then, the value $\sum_{i=0}^{h-2} d_i(d_{i+1} - 1)$ is at most $\sum_{i=0}^{h'-2} d'_i(d'_{i+1} - 1)$.*

Proof. Let $D = (d_0 = n, d_1, \ldots, d_h)$ and $D' = (d'_0 = n, d'_1, \ldots, d'_{h'})$ be two sequences of divisors of n such that $D <_{lex} D'$. We may further assume that $d_1 < d'_1$ (otherwise, $d_1 = d'_1$ and we can truncate both sequences, removing n, and see them as sequences of divisors of d_1 instead). Since d_1 and d'_1 are both divisors of n, then they are of the form $d_1 = n/a_1$ and $d'_1 = n/a'_1$ with $a'_1 < a_1$. Now, notice that, since D is a sequence of strictly decreasing divisors of n, we have

$$d_i \leq \frac{d_{i-1}}{2} \leq \frac{d_{i-2}}{2^2} \leq \ldots \leq \frac{d_1}{2^{i-1}} = \frac{n}{a_1 2^{i-1}},$$

for every $i \in \{1, \ldots, h\}$. Thus,

$$\begin{aligned} \sum_{i=0}^{h-2} d_i(d_{i+1} - 1) &= n(d_1 - 1) + \sum_{i=1}^{h-2} d_i(d_{i+1} - 1) \\ &\leq n(d_1 - 1) + \sum_{i=1}^{h-2} \frac{n}{a_1 2^{i-1}} \left(\frac{d_1}{2^i} - 1 \right) \\ &\leq n(d_1 - 1) + \sum_{i=1}^{h-2} \frac{n}{a_1 2^{i-1}} \left(\frac{d_1 - 1}{2^i} \right) \\ &\leq n(d_1 - 1) + \frac{n}{a_1}(d_1 - 1) \sum_{i=1}^{h-2} \frac{1}{2^{2i-1}}. \end{aligned}$$

However, $\sum_{i \in \mathbb{N}} 1/2^{2i-1}$ is a convergent series bounded above by 1. Therefore,

$$\sum_{i=0}^{h-2} d_i(d_{i+1} - 1) \leq n(d_1 - 1) + \frac{n}{a_1}(d_1 - 1) = n(d_1 - 1)\left(1 + \frac{1}{a_1}\right). \tag{3.4}$$

We now prove the inequality

$$n(d_1 - 1)\left(1 + \frac{1}{a_1}\right) \leq n(d'_1 - 1), \tag{3.5}$$

which is equivalent to proving that

$$d_1 + \frac{d_1}{a_1} - \frac{1}{a_1} \le d_1'.$$

We prove a slightly stronger inequality, that is, we have the following inequality:

$$d_1 + \frac{d_1}{a_1} \le d_1' \iff \frac{n}{a_1} + \frac{n}{a_1^2} \le \frac{n}{a_1'} \iff \frac{a_1+1}{a_1^2} \le \frac{1}{a_1'} \iff a_1'(a_1+1) \le a_1^2.$$

However remember that $a_1' < a_1$. Hence, $a_1'(a_1+1) \le (a_1-1)(a_1+1) = a_1^2 - 1 \le a_1^2$, so the inequality (3.5) follows. If we plug this in to the inequality (3.4), we obtain

$$\sum_{i=0}^{h-2} d_i(d_{i+1}-1) \le n(d_1'-1) \le n(d_1'-1) + \sum_{i=1}^{h'-2} d_i'(d_{i+1}'-1) = \sum_{i=0}^{h'-2} d_i'(d_{i+1}'-1),$$

as we wanted to show. □

Table 3 illustrates Proposition 7 for $n = 105$.

Table 3. List of sequences of divisors for degree 105 and minimum conductor of the A-M semigroup associated.

Sequence of divisors	Minimum conductor
$(105, 1, 0, 0)$	0
$(105, 3, 1, 0)$	210
$(105, 5, 1, 0)$	420
$(105, 7, 1, 0)$	630
$(105, 15, 1, 0)$	1470
$(105, 15, 3, 1)$	1500
$(105, 15, 5, 1)$	1530
$(105, 21, 1, 0)$	2100
$(105, 21, 3, 1)$	2142
$(105, 21, 7, 1)$	2226
$(105, 35, 1, 0)$	3570
$(105, 35, 5, 1)$	3710
$(105, 35, 7, 1)$	3780

4. Conductors of A-M semigroups of even degree

Let $\mathcal{F}_n$ be the set of Abhyankar-Moh semigroups of degree $n > 2$. Let $\mathcal{E}_n = [n-1, (n-1)(n-2)] \cap 2\mathbb{Z}$. The cardinality of $\mathcal{E}_n$ is

$$\sharp\mathcal{E}_n = \begin{cases} \frac{(n-1)(n-3)+2}{2} & \text{when } n \text{ is odd} \\ \frac{(n-1)(n-3)+1}{2} & \text{when } n \text{ is even.} \end{cases} \tag{4.1}$$

For any $c \in \mathcal{E}_n$, is there an $S \in \mathcal{F}_n$ such that $c(S) = c$? We will prove that the answer is positive when n is even. However, if n is odd, then this is not true. Indeed, suppose that S is an A-M semigroup of a prime degree $n > 2$. In this case, $S = S(v_0, v_1)$, where $n = v_0$ is greater than v_1, $v_1 > 1$, and $\gcd(v_0, v_1) = 1$. Hence,

$$\mathcal{F}_n = \{(n, v_1) \ : \ 2 \le v_1 \le n-1\},$$

and the cardinality of $\mathcal{F}_n$ is $n-2$. By (4.1), the cardinality of $\mathcal{E}_n$ is $\frac{(n-1)(n-3)}{2} + 1$. Observe that $\frac{(n-1)(n-3)}{2} + 1 > n-2$ for any $n > 3$. So, for $n > 3$, we conclude that there are values in $\mathcal{E}_n$ which are not realizable as conductors of an A-M semigroup of degree n. Remember that the only A-M semigroup of degree 2 is $\mathbb{N}$. Suppose that n is an even integer greater than 2. First, we will prove that for any $c \in [n-1, \frac{n^2-2}{2}) \cap 2\mathbb{N}$ there is an A-M semigroup of degree n and conductor c.

Lemma 8. *Let $n \ge 4$ be an even integer. The only A-M semigroup S of degree n with $\mathrm{c}(S) = n$ is the semigroup generated by the characteristic sequence $(n, 2, n+1)$.*

Proof. The characteristic sequence $(n, 2, n+1)$ determines an A-M semigroup with conductor n. Let us prove the uniqueness. Let S be an A-M semigroup of degree $n = \mathrm{c}(S)$ determined by the characteristic sequence $(v_0 = n, v_1, v_2, \ldots, v_h)$. By [5, Proposition 1.2], $\mathrm{c}(S) = \sum_{i=1}^{h}(n_i - 1)v_i - v_0 + 1$ with $n_i > 1$ for $1 \le i \le h$. Suppose that $h \ge 3$. Since $v_1 < n < v_2 < v_3 < \cdots < v_h$, we get $\mathrm{c}(S) \ge \sum_{i=1}^{3}(n_i - 1)v_i - n + 1 > n$. On the other hand, if $h = 1$, then $c(S) = nv_1 - n - v_1 + 1$. Since $n > v_1 \ge 3$, $\mathrm{c}(S) \ge 3n - n - v_1 + 1 > n + 1$.

Let us suppose that $h = 2$, so $e_2 = 1$. Since $v_2 \ge n+1$, $\mathrm{c}(S) > (n_1 - 1)v_1 + (n_2 - 2)n$. Observe that for $n_2 > 2$ we get $\mathrm{c}(S) > n$. So, $n_2 = e_1 = 2$, $n_1 = n/2$, and v_1 is an even integer. If $v_1 \ge 4$ then $\mathrm{c}(S) \ge 2n - 2 > n$. Hence, $v_1 = 2$, and consequently, $\mathrm{c}(S) = n$ if and only if $v_2 = n+1$. □

Lemma 9. *Let $n \ge 4$ be an even integer. For any $q \in [0, \frac{(n-1)(n-2)+n}{4} - 1)$, the semigroup generated by $(n, 2, n+1+2q)$ is an A-M semigroup of conductor $n + 2q$.*

Proof. The characteristic sequence $(n, 2, n+1+2q)$ determines an A-M semigroup for any even integer $q < \frac{(n-1)(n-2)+n}{4} - 1$ with sequence of divisors $(n, 2, 1)$ and $\mathrm{c}(S(n, 2, n+1+2q)) = n + 2q$. □

Corollary 10. *For any even integer $n \ge 4$ and $c \in [n-1, \frac{n^2-2}{2}) \cap 2\mathbb{N}$, there is at least one A-M semigroup of degree n and conductor c.*

Now, the target will be to prove that, if $n \ge 4$ is an even integer and $c \in [\frac{n^2-2}{2}, (n-1)(n-2)] \cap 2\mathbb{N}$, there is an A-M semigroup of degree n and conductor c. We will do it in several steps.

Lemma 11. *Let n be a natural number such that $n = 2r \ge 4$, where r is odd. Let $k \in \mathbb{N}$ be co-prime with r and $1 \le k \le r-1$. The sequence $(n, 2k, v_2)$ determines an A-M semigroup of degree n if and only if it satisfies the following conditions:*

1. *v_2 is an odd number,*
2. *$kn + 1 \le v_2 \le 2r^2 - 1$.*

Proof. It follows directly from the definition of an A-M semigroup. □

Suppose that $(n = 2r, 2k, v_2)$ defines an A-M semigroup of degree n, where $1 \le k \le r-1$ with $\gcd(r, k) = 1$, so $e_0 = n = 2r > e_1 = 2 > e_2 = 1$. Denote this semigroup by $\mathcal{S}(k, v_2)$. The conductor of $\mathcal{S}(k, v_2)$ equals $(r-1)2k + v_2 - 2r + 1$. Hence, if $\mathcal{S}(k, v_2)$ and $\mathcal{S}(k, v_2')$ are two A-M semigroups of degree n with $v_2 < v_2'$, then $\mathrm{c}(\mathcal{S}(k, v_2)) < \mathrm{c}(\mathcal{S}(k, v_2'))$. If we fix k, varying v_2 we get A-M semigroups attaining the following values for their conductor:

$$\mathrm{c}(\mathcal{S}(k, v_2)) \in [2kn - 2(k+r) + 2, kn - 2(k+r) + 2r^2] \cap 2\mathbb{N}.$$

Define $I_l := [2ln - 2(l+r) + 2, ln - 2(l+r) + 2r^2] \cap 2\mathbb{N}$ for any $1 \le l \le r-1$. Note that for any $l_1, l_2 \in \mathbb{N}$ with $l_1 < l_2$, $I_{l_1} \cap I_{l_2}$ is the empty set if and only if $l_2 \ge \frac{r^2+l_1r-l_1}{2r-1}$. Since $\frac{r^2+l_1r-l_1}{2r-1} > \frac{r+l_1}{2}$, we obtain that $I_{l_1} \cap I_{l_2} \neq \emptyset$ for every $l_1 \in \mathbb{N}$ and $l_2 \in (l_1, \frac{r+l_1}{2}] \cap \mathbb{N}$. We want to show that by also varying the values of k, we construct A-M semigroups covering all possible conductors. For that, we prove the following lemma.

Lemma 12. *Let $r, l \in \mathbb{N}$ where r is odd, $l \le r-2$ and $\gcd(l, r) = 1$. There is a $k \in \left(l, \frac{l+r}{2}\right] \cap \mathbb{N}$ coprime with r.*

Proof. Let $m := \min\left\{n \in \mathbb{N} \ : \ r - 2^n \le \frac{l+r}{2}\right\}$. Notice that such an integer exists since $l \le r-2$ by hypothesis. We claim that $k := r - 2^m \in \left(l, \frac{l+r}{2}\right]$ and is coprime with r. Indeed, $\gcd(r - 2^m, r) = \gcd(2^m, r) = 1$, due to r being odd. Moreover, by definition of m, $k \le \frac{l+r}{2}$. Assume by contradiction that $k = r - 2^m \le l$ and then $2r - 2^m \le l + r$, that is, $r - 2^{m-1} \le \frac{l+r}{2}$, which contradicts the minimality of m. Hence, $k \in \left(l, \frac{l+r}{2}\right] \cap \mathbb{N}$ is as desired. □

Therefore, we conclude the following:

Proposition 13. *Let n be a natural number such that $n = 2r$, where $r > 1$ is odd. For any even number c with $n-1 \le c \le (n-1)(n-2)$ there is an A-M semigroup of degree n whose conductor is equal to c.*

Proof. It is enough to consider the characteristic sequences $(n = 2r, 2k, v_2)$ defining A-M semigroups. Indeed, take $L := \{l \in [1, r-1] \cap \mathbb{N} \mid \gcd(l, r) = 1\} = \{l_1, l_2, \ldots, l_s\}$, with $l_t < l_{t+1}$ for any $t \in \{1, \ldots, s\}$. Lemma 11 and its following discussion tell us that, by considering these characteristic sequences, we construct A-M semigroups whose conductors cover all the values in $I_{l_1} \cup \ldots \cup I_{l_s}$. Moreover, by Lemma 12, we know that $I_{l_i} \cap I_{l_{i+1}} \neq \emptyset$ for every $i \in \{1, \ldots, s-1\}$. Hence, we cover all even integers from the minimum value in I_{l_1} to the maximum value in I_{l_s}. Since $l_1 = 1$ and $l_s = r-1$, these values are n and $(n-1)(n-2)$, respectively. This concludes the proof. □

Remark 14. *After Algorithm 1, we get that the A-M semigroups of degree 4 are $S_1 = S(4, 2, 5)$, $S_2 = S(4, 3)$ and $S_3 = S(4, 2, 7)$, where $c(S_1) = 4$ and $c(S_2) = c(S_3) = 6$. Hence, any $c \in \mathcal{E}_4$ is the conductor of an A-M semigroup of degree 4. Observe that the sequence of divisors of S_1 and S_3 is $(4, 2, 1)$, and the sequence of divisors of S_2 is $(4, 1)$.*

Proposition 15. *Let n be a natural number such that $n = 2^k r$, where $k \ge 1$, and $r > 1$ is odd. For any even number c with $n-1 \le c \le (n-1)(n-2)$ there is an A-M semigroup of degree n whose conductor is equal to c.*

Proof. We use induction over k. Notice that for $k = 1$, the statement holds by Proposition 13.

Let us assume that $k \ge 2$ and that the proposition holds for $k-1$. In addition, Remark 14 already covers the case $n = 4$, and thus we can further assume that $n \ge 8$.

We first observe that, by Corollary 10, for every even $c \in I_1 := [n-1, \frac{n^2-2}{2})$, there exists an A-M semigroup of degree n and conductor c. Thus, it remains to prove that the same is true for any $c \in \left[\frac{n^2-2}{2}, (n-1)(n-2)\right]$.

On the one hand, as explained in Lemma 9, $S' = \langle \frac{n}{2}, 2, \frac{n}{2} + 1 \rangle$ is an A-M semigroup with conductor $c(S') = \frac{n}{2}$. One can check that the gluing $S = S' \bigoplus_{2,d} \mathbb{N} = \langle n, 4, n+2, f \rangle$ is an A-M semigroup for every odd number $f \in \left[2n + 5, \frac{n^2}{2} - 1\right]$. Furthermore, (3.1) tells us that

$$c(S) = 2\frac{n}{2} + f - 1 = n + f - 1.$$

Hence, through this gluing we have shown that for any even number c belonging to $I_2 := \left[3n+4, \frac{n^2}{2}+n-2\right]$, the semigroup $S = \langle n, 4, n+2, c+1-n\rangle$ is an A-M semigroup with conductor c.

On the other hand, by the induction hypothesis, for $m = \frac{n}{2} = 2^{k-1}r$, we can guarantee that, for every even number $c' \in [m-1, (m-1)(m-2)]$, there exists an A-M semigroup $S'_{c'}$ of degree m and conductor c'. It is not hard to check that the gluing

$$S(c', f_i) = S'_{c'} \oplus_{2,f_i} \mathbb{N}$$

is an A-M semigroup of degree $2m = n$, for both values: $f_1 = \frac{n^2}{2} - 1$ and $f_2 = \frac{n^2}{2} - 3$. In addition, again by (3.1), the conductor $c(c', f_i)$ of $S(c', f_i)$ is

$$c(c', f_i) = 2c' + f_i - 1.$$

Now, notice the relations

$$c(c', f_2) = c(c', f_1) - 2$$

and

$$c(c'-2, f_1) = c(c', f_2) - 2.$$

These imply that, by making this gluing and recursively decreasing the value of c', alternating it with both f_1 and f_2, we can assure the existence of an A-M semigroup of degree n and conductor c, for every even number from the maximum possible value, that is,

$$c((m-1)(m-2), f_1) = 2\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right) + \frac{n^2}{2} - 2 = (n-1)(n-2),$$

and all the way down to

$$c(m, f_2) = 2\frac{n}{2} + \frac{n^2}{2} - 3 - 1 = \frac{n^2}{2} + n - 4.$$

Overall, we have shown that, for any even number $c \in I_1 \cup I_2 \cup I_3$ there exists an A-M semigroup of degree n and conductor c, where $I_3 = \left[\frac{n^2}{2} + n - 4, (n-1)(n-2)\right]$. Clearly, I_2 and I_3 overlap, and we have that $I_1 \cup I_2 \cup I_3 = [n-1, (n-1)(n-2)]$ as long as $n \geq 8$. Thus, the statement follows. □

Let $p > 1$ be a prime number and k be an integer greater than or equal to two. Note that, for every A-M semigroup S of degree $n = p^k$, $\mathrm{c}(S)$ is a multiple of $p-1$ (see equation (2.1)). So, $\mathrm{c}(S) \in [n-1, (n-1)(n-2)] \cap (p-1)\mathbb{N}$.

Now, we will study the A-M semigroups of degree $n = p^k$, where p is a prime number and $k \in \mathbb{N}$, $k > 1$. For any natural number k_1 with $1 \leq k_1 \leq p^{k-1} - 1$, we define $I_{k_1} := [1, p^{2k-1} - p^k k_1 - 1]$, and

$$\mathcal{A}_{k_1} := \left\{(p^k - 1)(pk_1 - 1) + i(p-1) \ : \ i \in I_{k_1} \text{ and } \gcd(p, i) = 1\right\}.$$

In the following proposition, we prove that for a fixed $n = 2^k$, every value in $[n-1, (n-1)(n-2)] \cap 2\mathbb{N}$ is the conductor of some A-M semigroup of degree n. Furthermore, the conductors of the A-M semigroups of degree $n = p^k$ are provided explicitly for any odd prime integer p, and the sequences of divisors of n that determine these conductors are made explicit.

Proposition 16. *Let $p \geq 2$ be a prime number, and $k \geq 2$ is an integer. Any conductor of an A-M semigroup with degree $n = p^k$ is obtained from, at most, two types of sequences of divisors:*

1. *For p, a prime odd number, the sequences are $(p^k, 1)$ and $(p^k, p, 1)$;*
2. *For $p = 2$ and $k = 2$, the sequence is $(4, 2, 1)$;*
3. *For $p = 2$ and $k = 3$, the sequences are $(8, 2, 1)$ and $(8, 4, 2, 1)$;*
4. *For $p = 2$ and $k \geq 4$, the sequence is $(2^k, 2, 1)$.*

Moreover, for any odd prime number p, the set of the conductors of A-M semigroups of degree n is

$$\left(\bigcup_{i=1,\, \gcd(p,i)=1}^{p^k-2} \{i(p^k - 1)\} \right) \bigcup \left(\bigcup_{k_1=1,\, \gcd(p,k_1)=1}^{p^{k-1}-1} \mathcal{A}_{k_1} \right).$$

Proof. Let (p^k, pk_1, v_2) be a characteristic sequence where $\gcd(p, k_1) = \gcd(p, v_2) = 1$. Note that the semigroup $S(p^k, pk_1, v_2)$ is an A-M semigroup if and only if $p^k k_1 < v_2 < p^{2k-1}$. So, we assume that $k_1 \in [1, p^{k-1} - 1]$ and $v_2 \in [p^k k_1 + 1, p^{2k-1} - 1]$, with k_1 and v_2 coprime with p. Consider $v_2 = p^k k_1 + i$ with $i \in I_{k_1} = [1, p^{2k-1} - p^k k_1 - 1]$ and such that $i \neq 0 \mod p$.

Thus, $\mathrm{c}(S) = (p^k - 1)(pk_1 - 1) + i(p - 1)$. Define $\mathrm{c}(p^k, pk_1, p^k k_1 + i) = (p^k - 1)(pk_1 - 1) + (p - 1)i$. This even value belongs to $C(k_1) \cap (p - 1)\mathbb{N}$, where $C(k_1) = [(p^k - 1)(pk_1 - 1) + (p - 1), (p^k - 1)(pk_1 - 1) + (p - 1)(p^{2k-1} - p^k k_1 - 1)]$, for any $i \in I_{k_1}$. Note that, for $p = 2$, $\mathrm{c}(2^k, 2k_1, 2^k k_1 + i)$ takes all the even numbers in $C(k_1)$. In addition, for any integer k_1' such that $k_1 < k_1'$, we have that

$$(C(k_1) \cup C(k_1')) \cap 2\mathbb{N} \cap (p - 1)\mathbb{N} = [\min C(k_1), \max C(k_1')] \cap 2\mathbb{N} \cap (p - 1)\mathbb{N} \tag{4.2}$$

if and only if $\min C(k_1') \leq \max C(k_1) + \max\{2, p - 1\}$.

Assume that $p \geq 3$ and suppose $\gcd(p, k_1 + 1) = 1$. In that case, the condition (4.2) holds for $k_1 + 1 \leq p^{k-1} - 1$ iff $\min C(k_1 + 1) \leq \max C(k_1) + p - 1$, that is, iff $(p - 1)(p^{2k-1} - p^k k_1 - 1) - p(p^k - 1) \geq 0$. Since $k_1 \leq p^{k-1} - 2$,

$$\begin{aligned} (p - 1)(p^{2k-1} - p^k k_1 - 1) - p(p^k - 1) &\geq (p - 1)(2p^k - 1) - p(p^k - 1) \\ &= p^k(p - 2) + 1 > 0. \end{aligned}$$

Thus, the condition (4.2) holds when $\gcd(p, k_1 + 1) = 1$. In the case where p and $k_1 + 1$ are not coprime, we have that $\gcd(p, k_1 + 2) = 1$, and the inequality $\min C(k_1 + 2) \leq \max C(k_1) + p - 1$ is equivalent to $(p-1)(p^{2k-1} - p^k k_1 - 1) - 2p(p^k - 1) \geq 0$. Again, since $k_1 + 2 \leq p^{k-1} - 1$, $(p-1)(p^{2k-1} - p^k k_1 - 1) - 2p(p^k - 1) \geq p^k(p - 3) + p + 1 > 0$. We conclude that, for $p \geq 3$, the condition (4.2) always holds. That means that, for every $k_1 \in [1, p^{k-1} - 1]$ and $i \in I_{k_1}$ with $i \mod p \neq 0$, $\mathrm{c}(p^k, pk_1, p^k k_1 + i) = (p^k - 1)(pk_1 - 1) + i(p - 1) \in \mathcal{A} = [\min C(1), \max C(p^{k-1} - 1)] \cap (p - 1)\mathbb{N} = [n(p - 1), (n - 1)(n - 2)] \cap (p - 1)\mathbb{N}$. Note that $\mathcal{A}$ is the union of the disjoint sets

$$\mathcal{A}_{k_1} = \bigcup_{k_1=1,\, \gcd(p,k_1)=1}^{p^{k-1}-1} \left\{ c(p^k, pk_1, p^k k_1 + i) \mid i \in I_{k_1}, \text{ and } \gcd(p, i) = 1 \right\},$$

and

$$\mathcal{A}'_{k_1} = \bigcup_{k_1=1,\, \gcd(p,k_1)=1}^{p^{k-1}-1} \left\{ c(p^k, pk_1, p^k k_1 + i) \mid i \in I_{k_1}, \text{ and } \gcd(p, i) \neq 1 \right\}.$$

The elements of $\mathcal{A}'_{k_1}$ are not the conductors of any A-M semigroup of degree n. Suppose there exists an A-M semigroup S' of degree n such that its conductor belongs to $\mathcal{A}'_{k_1}$. If the characteristic sequence of S' is (p^k, v_1) with $\gcd(p, v_1) = 1$, then $c(S') = (p^k - 1)(v_1 - 1) = (p^k - 1)(pk_1 - 1) + i(p - 1)$, and $(p^k - 1)v_1 = (p^k - 1)pk_1 + i(p - 1)$. However this is not possible since $\gcd(p^k - 1, p) = \gcd(p, v_1) = 1$. If the length of the characteristic sequence of S' is greater than or equal to three, by Proposition 3, there exists an A-M semigroup S'', $q \in \mathbb{N}$ and $f \in \mathbb{N}$ with $1 < q < k$ and $\gcd(p, f) = 1$, such that $S' = S'' \oplus_{p^q,f} \mathbb{N}$. Thus, by equality (3.1), $c(S') = p^q c(S'') + (p^q - 1)(f - 1) = (p^k - 1)(pk_1 - 1) + i(p - 1)$, and $(p^q - 1)f = (p^k - 1)pk_1 + i(p - 1) - p^q c(S'') - p^q(p^{k-q} - 1)$. Since $\gcd(p^q - 1, p) = \gcd(p, f) = 1$, it does not hold.

Note that, by Proposition 3 and the equality (3.1), $n(p - 1)$ is the smallest conductor of an A-M semigroup of degree n that can be obtained from characteristic sequences of length greater than or equal to three. Moreover, the set of the conductors of the A-M semigroups of degree n with a characteristic sequence of length two is $\cup_{v_1=2,\, \gcd(p,v_1)=1}^{p^k-1}\{(v_1 - 1)(p^k - 1)\}$.

Summarizing, we have just proved that, for every prime odd number $p \geq 3$ and $k \geq 2$, the conductor of any A-M semigroup with degree $n = p^k$ is equal to $i(p^k - 1)$ with $i \in [1, p^k - 2]$ satisfying $i \mod p \neq 0$, or it is equal to $(p^k - 1)(pk_1 - 1) + i(p - 1)$ where $k_1 \in [1, p^{k-1} - 1]$ and $i \in I_{k_1}$ such that $i \mod p \neq 0$. For the first type, their sequences of divisors are $(p^k, 1)$, and we have $(p^k, p, 1)$ for the second one.

For $p = 2$, $\gcd(2, k_1 + 1) \neq 1$, but $\gcd(2, k_1 + 2) = 1$. Hence, the condition (4.2) holds if and only if $2^{2k-1} - 2k_1 - 2^{k+2} + 3 \geq 0$. Using the upper bound of k_1, $2^{2k-1} - 2k_1 - 2^{k+2} + 3 \geq 2^{2k-1} - 5 \cdot 2^k + 5$. It is easy to prove that $2^{2k-1} - 5 \cdot 2^k + 5 \geq 0$ for any $k \geq 4$. Therefore, for $k \geq 4$, every even number in $[n - 1, (n - 1)(n - 2)]$ is realizable as the conductor of an A-M semigroup with the sequence of divisors $(2^k, 2, 1)$. The particular cases $k = 2$, and $k = 3$ are showed in Remark 14 and Table 2, respectively. □

By Proposition 15 and Proposition 16, for $p = 2$, we have the following,

Theorem A. *Let $n > 2$ be an even natural number. For any even number c with $n - 1 \leq c \leq (n - 1)(n - 2)$, there is an A-M semigroup of degree n and conductor equal to c.*

Open question: Characterize the values of conductors of A-M semigroups of odd degree. Note that Proposition 16 solves this question for degree p^k with p any prime integer.

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Conflict of interest

The authors declare there are no conflicts of interest.

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